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Spacetime Topology (I) – Chirality and the Third Stiefel–Whitney Class

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The standard model of particle physics poses certain limitations upon the topology of spacetime, most notably by imposing the triviality of an important family of characteristic classes, the Stiefel–Whitney classes. In this, the first of two articles, we present a physical interpretation of the first three Stiefel–Whitney classes. While the relationship of the first two to the existence of spinor fields has been known since the sixties, apparently no connection between the third class and microscopical physics seems to be known. We show that the third class is related to chirality.

1. INTRODUCTION

At first sight, it might be surprising that microscopical physics puts limitations on the global topological structure of spacetime, yet it is known that the first two so-called Stiefel–Whitney classes, w_1 and w_2 (the generators of the Čech cohomology groups $\check{H}^1(M, \mathbb{Z}_2)$ and $\check{H}^2(M, \mathbb{Z}_2)$ respectively, see appendix), have far-reaching consequences for physics, but this far the third and fourth remain to be investigated.

We will give a short introduction to this result, and then concentrate on the next Stiefel–Whitney class. When we wish to construct a *physically reasonable* spacetime to investigate, for example, global causality features, we should construct the spacetime model so that it does not violate any deeply held physical principles. But we should also investigate if the mathematics we employ may not give us further information. It is the purpose of this, the first of two articles by the authors, to prove that the third Stiefel–Whitney class does have a physical interpretation. In fact, we claim that w_3 is related to chirality. The triviality of w_1 leads to spacetime being orientable, when w_2 is trivial it is possible to erect spinor

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bundles and when both are trivial then obstruction theory from algebraic geometry tells us that w_3 is trivial, too (Bredon, 1993).

Obviously any physically reasonable spacetime must have $w_1 = w_2 = 1$ and therefore w_3 trivial by construction. However, it is our conjecture that all four Stiefel–Whitney classes of a physically reasonable spacetime must be trivial, and all have physical significance. It is the purpose of this article to review the first two in a physical context and show the physical significance of the third.

We will begin by listing some immediate topological restrictions to be imposed on any candidate for a physical spacetime manifold. Next, we will outline the importance of the first two Stiefel–Whitney classes, before we attempt a generalization to w_3. Furthermore, we will suggest a natural chain of groups to be studied in this respect. All of our results are, unless explicitly stated otherwise, restricted to four dimensions.

Throughout this paper, we will try to avoid a heavy use of homological algebra and instead concentrate on simple physical ideas. The paper can thus be seen as a physicist's view of the Stiefel–Whitney classes.

The notation is fairly standard: de Rham cohomology is denoted by H^r, while Čech cohomology is denoted by $\check{H}^r$, M denotes a four-dimensional spacetime of Lorentzian signature, $\Omega^p(M)$ the space of p forms on M, $\mathcal{F}(M)$ the space of real functions $f : M \to \mathbb{R}$, while Γ denotes the set of sections in a given bundle.

2. A PHYSICALLY REASONABLE SPACETIME IN GENERAL RELATIVITY

General relativity tells us that spacetime, M, is a four-dimensional smooth manifold. It is equipped with a Lorentz (pseudo-Riemannian of index (1, 3)) metric that divides the elements of each tangent space $T_m M$ into three classes, timelike, null-like, and spacelike. A chart on M is a pair (U_i, φ_i) of a coordinate neighborhood U_i and a coordinate function φ_i, so that any event $m \in M$ can be expressed as $\varphi_i(m) = \{x^0(m), x^1(m), x^2(m), x^3(m)\}$. Without loss of generality we can take M to be also paracompact, since a manifold M is known to admit a Lorentzian metric if and only if it is paracompact (and it admits an everywhere-nonvanishing continuous direction field) (Visser, 1996).

We shall employ the following groups:

$O(1, 3) = \{A \in GL(4, \mathbb{R}) \mid A\eta A^t = \eta\}$, the Lorentz group

$SO(1, 3)^+ = \{A \in O(3, 1) \mid \det A = 1, A_0^0 > 0\}$, the proper orthochronous Lorentz group

$SL(2, \mathbb{C}) = \{A \in GL(4, \mathbb{C}) \mid \det A = 1\}$, the special linear group

$SU(2) = \{A \in GL(4, \mathbb{C}) \mid AA^\dagger = A^\dagger A = 1, \det A = 1\}$, the special unimodular group

In other words $SO(1, 3)^+$ is the set of $A \in SO(1, 3)$ with A_{00} positive (i.e. these $\{A\}$ do not change the sign of time), and in the literature the notation $\mathcal{L}_0^{\uparrow}$ is also often used. We have chosen $SO(1, 3)^+$ for clarity.

We usually reject spacetimes having closed timelike curves and possibly also incomplete geodesics, but that is not to say that it has been proven that they do not exist in a spacetime that as accurately as possible describes the universe we live in. Therefore we will not make any assumptions that equip M with causality features we cannot prove. We will, however, reject conditions on M that are known to *guarantee* the existence of closed timelike curves. What we will assume is that as Hawking and Ellis (1973) write, "in order to be physically significant, a property of space-time ought to have some form of stability, that is to say, it should also be a property of 'nearby' space-times."

We can now ask if the mathematical features of our spacetime model violate any deeply held physical principles.

2.1. Compactness of *M*

Paracompact is a rather weak mathematical requirement to impose on a spacetime manifold. Would it be physically reasonable to sharpen it to assume M to be compact, or can we only assume that M is noncompact, although paracompact? It is known (Hawking and Ellis, 1973) that a compact Lorentzian manifold contains closed timelike curves and would hence have problems with causality. Thus we cannot take M to be compact.

Now, with M noncompact, we have to be careful, as many results in the mathematical literature only hold for compact manifolds. Since M carries a metric it is triangulizable (Nakahara, 1990). We can then define the rth Betti numbers as the rank of the free Abelian part of the rth homology group

$$b_r(M) = \dim H_r(M; \mathbb{R})$$

and the Euler characteristic as

$$\chi(M) = b_0 - b_1 + b_2 - b_3 + b_4.$$

But we do not have Poincaré duality, since M is only assumed to be paracompact, unless all the (co)homology groups are finite (Goldberg, 1962).

2.2. Connectedness and the de Rham Cohomology

As a topological space M is connected if it cannot be written as the disjoint union $M = M_1 \cup M_2$ of open sets M_1 and M_2, $M_1 \cap M_2 = \emptyset$. If the universe truly consisted of two disjoint subuniverses that cannot interact, it is reasonable to say that *our* universe only consists of the subuniverse that we reside in. We can therefore claim that a physically reasonable universe is connected, which tells us that the zeroth de Rham cohomology group is the set of real numbers $H^0(M; \mathbb{R}) = \mathbb{R}$. We can further assume that M is also archwise connected, giving us that $H_0(M; \mathbb{R}) = \mathbb{Z}$ and $b_0 = 1$.

Is it reasonable to assume the stronger demand, namely that all loops in M can be shrunk to a point, or that M is simply connected? Since we do not want to exclude the existence of exotic structures such as wormholes, we cannot impose the vanishing of the fundamental group $\pi_1(M)$—clearly, a closed closed curve going through a wormhole cannot be shrunk to a point. Thus spacetime may be taken to be archwise connected but need not be simply connected.

2.3. Orientability of *M*

To progress further in imposing constraints on M, we have to move beyond general relativity. Through the Einstein equations, the matter content of the universe imposes constraints upon its geometry. We will see that the known matter fields also impose topological conditions. It is known that the experimental evidence of nonconservation of C, P, and CP in elementary particle reactions, the *CPT* theorem, and the strong principle of equivalence together imply that our universe must be orientable (Visser, 1996).

Theorem 1. *Nonorientable spacetimes are incompatible with the standard model of particle physics.*

2.4. Causality Features of *M*

We observe in our local region of spacetime that from any one event m we are only able to influence those events that lie in the *forward lightcone* of m. But whether or not this is a global property remains to be investigated, and this will be done in our next article.

Aside from the gravitational field g there will be various other fields on M, such as the electromagnetic field, the neutrino field, etc. Mathematically, fields are sections of certain fibre bundles over spacetime. The gauge fields will be connections on principal bundles. The equations of motion governing the matter fields must be such that if U is a convex normal neighborhood and m and m' are points in U, then a signal can be sent in U between m and m' if and only if m and m' can be joined by a C^1 (differentiable) curve lying entirely in U, whose tangent vector is everywhere nonzero and is either timelike or null; we shall call such a curve *nonspacelike*. This is the *local causality condition* (Hawking and Ellis, 1973).

Whether the signal is sent from m to m' or from m' to m will depend on the direction of time in U. In our neighborhood of spacetime there is a well-defined arrow of time given by the direction of increase of entropy in quasi-isolated thermodynamic systems, making it possible to distinguish past and future at least locally. But it suffices to observe that the microphysics of the weak interactions experimentally breaks time-reversal invariance (that is, the T of the CPT theorem); furthermore, we cannot even begin to define time reversal in time-nonorientable manifolds.

Theorem 2. *Time-nonorientability spacetimes are incompatible with the standard model of particle physics.*

Physically it would seem reasonable to suppose that there is a local thermodynamic arrow of time defined continuously at every point of spacetime, but we shall only require that it should be possible to define continuously a division of nonspacelike vectors into two classes, which we label future- and past-directed. This means that M is *time-orientable*, which is indeed a physically reasonable requirement since M has a Lorentzian metric (Visser, 1996). We can now ask if M is also *space-orientable*, that is, if it is possible to divide bases of three spacelike axes into right-handed and left-handed bases in a continuous manner.

Definition 1. A spacetime is space-orientable if (1) there exists a continuous and everywhere-nonzero globally defined three-form $\omega = \frac{1}{6}\omega_{\nu\mu\lambda} dx^\nu \wedge dx^\mu \wedge dx^\lambda$, and (2) there exists an everywhere-nonzero timelike vector field that is continuous up to possible sign reversal, known as the direction field $d(m)$, and (3) $\omega(d) = 0$.

We observe, locally, the ability to distinguish left from right, and in microphysics we also observe the breakdown of parity invariance (the P of the CPT theorem). If we believe that the noninvariance of weak interactions under charge and parity reversals is not merely a local effect but exists at all points of spacetime, then it follows that going around any closed curve the sign of a charge, the orientation of a basis of spacelike, axes, and the orientation of time must all reverse or none of them should. So if one assumes that spacetime is time-orientable then it must also be space-orientable (Visser, 1996).

Theorem 3. *Space-nonorientability spacetimes are incompatible with the standard model of particle physics.*

The assumption that our four-dimensional manifold M carries a $(1 + 3)$ signature and is noncompact has three immediate consequences.

One is that the Dirac operator is no longer elliptic (Gockler and Schucker, 1987), $\gamma_5^2 = -I$ and M does not a priori contain closed timelike curves. We want our model to be in agreement with findings in modern physics, be it the standard model, general relativity, or cosmology. This already gives us that $w_1 = w_2 = 1$

and we will begin with reviewing in details the first and second Stiefel–Whitney classes, and collect the consequences of the triviality of both.

3. SPACETIME ORIENTABILITY AND THE FIRST STIEFEL–WHITNEY CLASS

An orientation on M can be defined using homology, forms, or the transition functions for the tangent bundle TM. For sake of completeness, we will list all these definitions here.

In (Milnor and Stasheff, xxxx) we find that a *local orientation* μ_m for M ($\dim M = n$) at m is a choice of one of the two possible generators for $H_n(M, M - m; \mathbb{Z})$[4]. Such a local orientation μ_m has the physical property that it determines local orientations $\mu_{m'}$ for all points m' in a small neighborhood of m. To see this define $\rho_K : H_i(M, M - L) \to H_i(M, M - K)$, where $K \subset L$ are both compact and contained in M. The image of ρ_K is thus a restriction to K. Let O be a ball around m, then for each $m' \in O$ the isomorphisms (Husemoller, 1994)

$$H_*(M, M - m; \mathbb{Z}) \overset{\rho m}{\leftarrow} H_*(M, M - O; \mathbb{Z}) \overset{\rho m'}{\rightarrow} H_*(M, M - m', \mathbb{Z})$$

determine a local orientation $\mu_{m'}$.

Definition 2. An *orientation* for M is a function that assigns to each $m \in M$ a local orientation μ_m that varies continuously with m in the following way: For each m there should exist a compact neighborhood N and a homology class $\mu_N \in H_n(M, M - N)$ so that $\rho_{m'}(\mu_N) = \mu_N$ for each $m' \in N$.

So in our case an orientation is a homology-valued function on M, $f : M \to H_4(M, M - m; \mathbb{Z})$.[5] An alternative definition of orientability is the following

Definition 3. If the tangent bundle $\pi : TM \to M$ is n-dimensional then $\Omega^n(M)$ is one-dimensional. This means that $\Omega^n(M) - \{0\}$ has two components. An *orientation* on TM (and on M) is a choice of one of the components of $\Omega^n(V) - \{0\}$.

[4] For the reader unfamiliar with homology theory it may help to think of a sphere around the removed point m. We can then define two inequivalent ways of going around the missing point, one for each generator.

[5] It is worth noting that if the subset K containing m is compact, then there is one and only one homology class $\mu_K \in H_n(M, M - K; \mathbb{Z})$ which satisfies $\rho_m(\mu_K) = \mu_m$ for each $m \in K$ (Milnor and Stasheff, 1974). If M itself is compact then there is one and only one μ_M with this property. This homology class $\mu = \mu_M = [M]$ is called the *fundamental homology class* of M. This does not mean that M compact possess only one generator of $H_n(M, M - m; \mathbb{Z})$. The two generators of this homology group can be thought of as the two inequivalent ways of going round a closed loop with the centre point removed. But in case M is compact only one of the generators satisfy the property mentioned above.

Any orientable manifold then admits two inequivalent orientations, often called right-handed and left-handed, respectively. The chosen n-form (orientation) vanishes nowhere and is called the *volume element*. It plays the role of a measure when we integrate functions $f \in \mathcal{F}(M)$, and integrations of differential forms over M is defined only when M is orientable.

For the reader unfamiliar with this use of forms, it can help to think of it in the following way: At a point $m \in M$ the tangent space $T_m M$ is spanned by the basis $\{e_\mu\} = \frac{\partial}{\partial x^\mu}$, where x^μ is the local coordinate on the chart U_i to which m belongs. If we let U_j be another chart such that $U_i \cap U_j \neq \emptyset$ with local coordinates y^μ and let $m \in U_i \cap U_j \neq \emptyset$, then $T_m M$ is spanned either by $\{e_\mu\}$ or by $\{\tilde{e}_\mu\} = \{\frac{\partial}{\partial y^\mu}\}$.

The basis changes as

$$\tilde{e}_\nu = \frac{\partial x^\mu}{\partial y^\nu}\, e_\mu,$$

where $(\partial x^\mu / \partial y^\nu)$ are known as the transitions functions t_{ij} in the theory of fibre bundles. If the Jacobian $J = \det(\partial x^\mu / \partial y^\nu) = \det(t_{ij}) > 0$ on $U_i \cap U_j$ then the two bases $\{e_\mu\}$ and $\{e_\nu\}$ are said to define the same orientation and M is said to be orientable.

Definition 4. The *orientation bundle* of M is the line bundle LM on M given by the transition functions $\tau_{ij} = \text{sign} \det(t_{ij})$.

Clearly, if M is orientable the orientation bundle is trivial, since the transitions functions all become identically one. Moreover, the transition functions have to satisfy the cocycle condition and hence form elements of the first Čech cohomology group, $\check{H}^1(M, \mathbb{Z}_2)$. The generator of this group, ω_1, is known as the *first Stiefel–Whitney class*. Thus $[\tau_{ij}] = w_1$, and this gives the relationship between orientability and triviality of the first Stiefel–Whitney class. For the particular case of a $(3+1)$-dimensional manifold, we can summarize all of this in a theorem.

Theorem 4. *For our four-dimensional spacetime manifold M the following statements are equivalent:*

1. *M is orientable (that is, spacetime-orientable, time-orientable, and space-orientable).*
2. *There is a collection $\Phi = \{(U, \varphi)\}$ of coordinate systems on M such that*

$$M = \cup_{(U,\varphi)\in\Phi} U \quad \textit{and} \quad \det\left(\frac{\partial x_\mu}{\partial y_\nu}\right) > 0 \quad \textit{on} \quad U_i \cap U_j$$

$$\textit{whenever } (U_i; x_0, \ldots, x_3) \textit{ and } (U_j; y_0, \ldots, y_3) \textit{ belongs to } \Phi.$$

In other words the Jacobian determinant for coordinate transformations on $U_i \cap U_j$ must be positive definite.

3. *The structure group of the tangent bundle reduces from* $O(1, 3)$ *to* $SO(1, 3)^+$.
4. *There is a nowhere-zero four-form on* M

$$\omega_4 = \omega_{0123}(m)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \in \Omega^4(M).$$

5. *The first Stiefel–Whitney class* $w_1(M)$ *is trivial.*
6. TM *is orientable.*
7. M*'s orientation bundle is trivial, that is it possesses a global section.*

For proof, see Geroch (1967, 1968, 1970) and Visser (1996). So as we incorporate the standard model of physics into our spacetime, we find that spacetime orientability is connected to the existence of a nowhere-vanishing section in $\Omega^4(M)$, the vanishing of the first Stiefel–Whitney class and the positive definiteness of the Jacobian matrix. In fact the Stiefel–Whitney classes that are elements of $\check{H}^r(M, \mathbb{Z}_2)$ are obstructions to global sections in certain bundles, as will become clear as we proceed. We refer the reader unfamiliar with Čech-cohomology and characteristic classes to Bott and Tu (1982) or the appendixes.

It is worth mentioning that it is because M is time-orientable (in addition to being spacetime-orientable) that the structure group reduces to the proper Lorentz group and not only to $SO(1, 3)$. This reduction of the structure group will be important for the existence of spinor structures in the next section and the triviality of the second Stiefel–Whitney class. Also note that since M is endowed with a metric we can define the *invariant volume element*, which is invariant under coordinate transformations, by

$$\Omega_4 = \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

where $g = \det g_{\mu\nu}$, and x^μ the coordinates of the chart (U, φ).

4. SPINOR STRUCTURE AND THE SECOND STIEFEL–WHITNEY CLASS

The next important ingredient in the standard model is the existence of families of fermions (leptons and quarks); thus our spacetime manifold must accommodate spinor fields. This turns out to impose yet another condition upon its topology. First, however, we need to quickly summarize what we mean by defining spinors on a general manifold.

Recall that for any n-dimensional vector space, such as $T_m M$ equipped with a metric g of signature $r + s = n$ we can define the *Clifford algebra* $C(r, s)$ as the real associative algebra generated by the elements of the vector space that satisfy

$$VW + WV = 2g(V, W)I.$$

In $n = 4$ dimensions our 16-dimensional Clifford algebra has the following direct sum compositions:

$$C(1,3) = \oplus_{i=0}^{4} C^i = \mathbb{R} \oplus T_m M \oplus \mathbb{C}^2 \oplus C^3 \oplus C^4. \tag{1}$$

We can then define the even subalgebra $C_e \equiv \oplus_{i\,\text{even}} C^i$ and the odd subspace $C_o \equiv \oplus_{i\,\text{odd}} C^i$. The elements of the vector space out of which we construct the Clifford algebra that are of unit length

$$g(V, V) = \pm 1$$

are made into the Clifford group Pin(r, s) generated by Clifford multiplication. Pin(r, s) is a submanifold of $C(r, s)$ and is therefore a Lie group with Lie algebra spin$(r, s) = C^2$, again contained in the Clifford algebra itself. The Special Clifford group, Spin(r, s), is the subgroup of Pin(r, s) consisting of even elements:

$$\text{Spin}(r, s) = \text{Pin}(r, s) \cap C_e.$$

For $n \geq 3$, Spin$(r, s)^+$ is simply connected, and since $SO(r, s)^+$ and Spin$(r, s)^+$ share the same Lie algebra, Spin$(r, s)^+$ is the universal covering group of the proper Lorentz group. Any representation of $SO(1, 3)^+$ then also yields a representation of its universal covering group, just by composition with the covering map φ. A spinor or double-valued representation of $SO(1, 3)^+$ is defined by a linear representation of Spin$(1, 3)^+ \cong SL(2, \mathbb{C})$ that cannot be obtained from a representation of $SO(3, 1)^+$.

Since the Clifford group Pin$(1, 3)$ is a subset of the Clifford algebra $C(1, 3)$ with the same product, the Dirac representation ρ of the algebra carried by $\mathbb{C}^4$ is at the same time a group representation. It is a spinor representation and the vectors of $\mathbb{C}^4$ are called Dirac spinors. But while the Dirac representation is irreducible as a representation of the Clifford algebra, it is *reducible* as a group representation.

M is said to admit a spin structure (which is not necessarily unique) if it is possible to define a *spin bundle* $S(M)$ over M. To define this bundle we first of all need to check if the frames used are globally defined. That is, we need a global section in the frame bundle FM associated with the tangent bundle TM. If x^μ are the coordinates on U_i and y^μ the coordinates on U_j, with $U_i \cap U_j \neq \emptyset$, the coordinate change when going from U_i to U_j can be written by means of elements from $SO(1, 3)^+$, the transition functions for TM (and by association, also for the frame bundle FM).

These transition functions satisfy

$$t_{ij} t_{jk} t_{ki} = 1, \quad t_{ii} = 1, \quad t_{ij} \in SO(3, 1)^+.$$

To define the spin-bundle we must check if $SO(1, 3)^+$ lifts to its universal covering group Spin$(1, 3)^+$, the Special Clifford Group using the double-cover homomorphism φ^{-1}, which takes its values in $\mathbb{Z}_2$.

For this “lifting” of $SO(1, 3)^+$ to exist, and thus define a spin bundle, the signs of ker$\varphi = \pm I$ must be patched together in a consistent manner. The obstruction to this is the second Stiefel–Whitney class w_2. Physically speaking, then, this characteristic class must be trivial, otherwise fermions would not be elementary particles.

A spin structure on M is then defined by the transition functions $\tilde{t}_{ij} \in \mathrm{Spin}(3, 1)^+ = SL(2, \mathbb{C})$ such that

$$\varphi(\tilde{t}_{ij}) = t_{ij}, \quad \tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = 1, \quad \tilde{t}_{ii} = 1.$$

A Dirac spinor is then a section of the the Dirac spin fibre bundle

$$(D(M), \pi, M, \mathbb{C}^4, SL(2, \mathbb{C}) \otimes \overline{SL(2, \mathbb{C})}),$$

that is, a $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ representation of the Lorentz group $SO(3, 1)^+$.[6] Since spinors are of such fundamental importance to physics it will be interesting to see what other properties are required of a manifold to allow a spinor structure. The existence of a spinor structure is first and foremost related to the topological properties of the underlying spacetime manifold M—not the choice of metric. But Geroch (1967) shows that the degree of curvature also must be investigated when one looks to see if spinor fields can be admitted. Since the Čech cohomology only deals with topology this is somewhat surprising. But for there to be even the possibility of a spacetime manifold not allowing spinor structure there must be a minimum amount of curvature. This curvature is expressed in the form of a surface integral and we have included a discussion of this in the appendixes.

Theorem 5. *For our paracompact four-dimensional manifold M the following holds* (Geroch, 1970):

1. *If the Weyl (or conformal) tensor in the Petrov classification (see appendixes) is everywhere,* [1, 1, 1, 1], [2, 1, 1], [3, 1] *or* [4], *or*
2. *if the Riemann tensor vanishes, or*
3. *if M arises from some initial-value data, that is if M has a Cauchy surface, then M possesses spinor structure.*

[6] Since we already saw in the previous section that we have time orientability, we can assume that the properness of the spin structures we will discuss below is implicit. For Lorentz signature manifolds we must usually distinguish between proper spin structure and spin structure. The former comes from the reduction of the Lorentz group $SO(1, 3)$ to the proper Lorentz group $SO(1, 3)^+$. A proper spin structure requires not only orientability but also time orientability for the Lorentz group to reduce to the proper. It is possible to define spinor structure without orientability, yet these alternative spinors all seem to be unphysical. If one wishes to define spinor fields with respect to the full Lorentz group the usual definition must be generalized. There are eight different simply connected covering groups of $SO(1, 3)$ that correspond to the various combinations of signs for P^2, T^2, and $(PT)^2$, where P (respectively T) is one of the two spin transformations corresponding to spatial reflections (time reflections). Hence one can consider eight types of spin structures. Spin(4), the full spine group is such that $P = \pm\varepsilon_0$ and $T = \pm\varepsilon^1\varepsilon^2\varepsilon^3$, such that $P^2 = T^2 = 1$ and $PTPT = -1$ (where ε^i are the Clifford generators.). Each situation would be handled differently, depending on the topology of the base space M and the structure desired for the vector bundle of spinors (DeWitt *et al.*, 1979).

For proof we refer the reader to Geroch (1967, 1968, 1970). If it is known that M possesses spinor structure we will however not generally know about the features of the Weyl or Riemann tensor. We can certainly not assume the Riemann curvature tensor to vanish, as this would imply a completely flat spacetime. But there are other ways of checking for the existence of a spinor structure.

Theorem 6. *For our paracompact four-dimensional oriented spacetime M the following statements are equal:*

1. *M possesses spinor structure.*
2. *M is parallelizable.*
3. *The second Stiefel–Whitney class w_2 is trivial.*
4. *M possesses a global section in the frame bundle FM.*
5. *The fundamental groups of the frame bundle FM and M are related as follows:*

$$\pi_1(FM) \approx \pi_1(M) \otimes \pi_1(F_m M) = \pi_1(M) \otimes \mathbb{Z}_2.$$

6. *The index $I(S)$ is an even integer for every two-surface S that is topologically a two-sphere, in M.*

For proof see, for example, Geroch (1967, 1968); for a definition of the index $I(M)$ see the Appendixes.

Most common solutions of Einstein's equations satisfy one or another of these conditions for M to have spinor structure (Geroch, 1970), but while certain of the above criteria can be easily checked, others cannot be. The Schwarzschild solution to the Einstein equations, for instance, does allow for a spinor structure, but it is not immediately clear that this spacetime has a nowhere-zero continuous section in its frame bundle FM, that is, a system of tetrads or vierbeins.

Note also, that statement 4 cannot in general by assumed to hold for other metric signatures and dimensions, while statement 2 only holds for a four-dimensional paracompact spacetime. Hence, as spacetime orientation was related to the vanishing of the first Stiefel–Whitney class and a nowhere-zero continuous section in $\Omega^4(M)$, so the existence of spinors is related to the vanishing of the second Stiefel–Whitney class, an element of $\check{H}^2(M, \mathbb{Z}_2)$ and a continuous global section in the frame bundle FM.

How do we determine if the spin-structure of M is unique? Each choice of spin structure determines an element of the first cohomology group $\check{H}^1(M, \mathbb{Z}_2)$ and, conversely, each element of $\check{H}^1(M, \mathbb{Z}_2)$ determines a spin structure on M, so that the set of spin structures is parametrized by $\check{H}^1(M, \mathbb{Z}_2)$. If $\check{H}^1(M, \mathbb{Z}_2)$ vanishes then the spinor structure of M is unique, which happens, for example, when M is simply connected (Geroch, 1968).

The triviality of the first two Stiefel–Whitney classes have, as we have seen, far-reaching consequences for physics. That abstract algebraic topology enters on such a fundamental level was what led us to start our research concerning the two remaining Stiefel–Whitney classes. $w_1 = 1$ leads to the reduction of the structure group from $O(1, 3)$ to $SO(1, 3)$ and then to $SO(1, 3)^+$, while the triviality of the second Stiefel–Whitney class makes it possible to lift the $SO(1, 3)^+$ bundle to the universal covering $SL(2, \mathbb{C})$ bundle over M. Both have $\mathbb{Z}_2$ as their kernels. We therefore propose the following chain of groups to be linked to the four Stiefel–Whitney classes:

$$O(1, 3) \to SO(1, 3) \to SO(1, 3)^+ \to SL(2, \mathbb{C}) \to SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})}$$
$$\to SU(2) \to SO(3).$$

$SO(3)$ and $SO(1, 3)^+$ are simple Lie groups; $SL(2, \mathbb{C})$ and $SU(2)$ are not. All but the last element in this chain of groups will be dealt with in this paper.

5. CHIRAL (WEYL) SPINORS AND THE THIRD STIEFEL–WHITNEY CLASS

Cohomology can be equipped with a special operation, the *Steenrod square*, $Sq^k : \check{H}^r(M, \mathbb{Z}_2) \to \check{H}^{r+k}(M, \mathbb{Z}_2)$. Since w_1 and w_2 are both trivial we can utilize the following theorem (Milnor and Stasheff, 1974):

Theorem 7. *The cohomology class $Sq^k(w_r)$ can be expressed as the polynomium*

$$\begin{aligned} Sq^k w_r &= \sum_{i=0}^{k} \binom{r-k+i-1}{i} w_{k-i} \smile w_{t+i} \\ &= w_k w_r + \binom{r-k}{1} w_{k-1} w_{r+1} + \binom{r-k+1}{2} w_{k-2} w_{r+2} \cdots \\ &\quad + \binom{r-1}{r} w_0 w_{r+k}, \end{aligned}$$

where $Sq^k : \check{H}^r(M, \mathbb{Z}_2) \to \check{H}^{r+k}(M, \mathbb{Z}_2)$.

This gives us

$$Sq^1 w_2 = w_1 w_2 + w_0 w_3, \tag{2}$$

and since w_0, w_1, and w_2 are all trivial we get that w_3, too, is a trivial cohomology class.[7] This, however, does not mean that w_3 in itself does not contain any interesting physical information. We will argue that w_3 is related to chirality. This is

[7] For more on Steenrod squares see Bredon (1993). It is an odd coincidence that this square can be used to calculate w_3, but not, e.g., w_4.

actually a rather natural question to study. We have seen that the first two Stiefel–Whitney classes are obstructions to the existence and uniqueness of a (proper) spin structure. Once spinors have been defined, it is natural to ask what *kinds* of spinors can be defined? Imposing no conditions, we of course get Dirac spinors, but we can try to impose extra conditions to see when Weyl or Majorana spinors can be defined. Majorana spinors are eigenstates of the charge conjugation operator, and they do not seem to exist in nature (Ramond, 1989). The neutral fermions, the neutrinos, that do exist are chiral, that is, have a definite handedness. A priori, the existence of chiral spinors poses further restrictions upon the spacetime manifold. Hence, that is what we are going to consider now.

Consider thus the Clifford algebra $C(3, 1)$ defined in the previous section. Since the element $\varepsilon^5 \equiv \varepsilon^0\varepsilon^1\varepsilon^2\varepsilon^3 \in C^4$ anticommutes with all elements of $C^1 = T_m M$ it commutes with all elements of C^2, since

$$[\varepsilon^5, \varepsilon^a\varepsilon^b] = \{\varepsilon^5, \varepsilon^a\}\varepsilon^b - \varepsilon^a\{\varepsilon^5, \varepsilon^b\} = 0, \tag{3}$$

where $\varepsilon^a\varepsilon^b$ are the six generators of C^2. By extension ε^5 then commutes with all of C_e so it belongs to the centre $Z(C_e)$ of this subalgebra, and can thus be diagonalized. Its action is then reduced to multiplication by a scalar (eigenvalue) and to each eigenvalue there corresponds an eigenspace. In other words we can use ε^5 to reduce the Dirac spinors.

For n even we define γ_5 as the Dirac representation of the generator of C^n:

$$\gamma_5 \equiv \rho(\varepsilon^0\varepsilon^1 \cdots \varepsilon^n). \tag{4}$$

With

$$\gamma_5^2 = (-1)^{\frac{n(n-1)}{2}+s} I,$$

where $r + s = n$ is the dimension and I is the $n \times n$ unit matrix; the following operators are projectors:

$$L \equiv \frac{1}{2}\left(I - \sqrt{(-1)^{\frac{n(n-1)}{2}+s}}\gamma_5\right) \tag{5}$$

$$R \equiv \frac{1}{2}\left(I + \sqrt{(-1)^{\frac{n(n-1)}{2}+s}}\gamma_5\right) \tag{6}$$

In our four-dimensional case with a (1, 3) signature we have

$$\gamma_5^2 = -I,$$

giving the projection operators the following form

$$L = \frac{1}{2}(I - i\gamma_5) \qquad R = \frac{1}{2}(I + i\gamma_5)$$

and a Dirac spinor $\psi \in \Gamma(M, (D, \pi_D, M, \mathbb{C}^4, SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})}))$ splits into two $SL(2, \mathbb{C})$-invariant components (in other words in addition to irreducible representations of the Clifford algebra $C(1, 3)$ we now also get irreducible representations of $\text{Spin}(1, 3)^+$)

$$\psi_L \equiv L\psi \qquad \psi_R \equiv R\psi$$

called left- and right-handed Weyl (or chiral) spinors.

The key factor to note about the standard model of particle physics is that the weak interactions are chiral, that is they exist in a definite eigenstate of the chiral operator γ_5:

$$\gamma_5\psi_L = -i\psi_L, \quad \text{and} \quad \gamma_5\psi_R = +i\psi_R$$

Particle physicists are used to seeing Eq. (4) in flat space defined as

$$\gamma_5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3 = \frac{\text{sign}[abcd]}{4!}\gamma^a\gamma^b\gamma^c\gamma^d,$$

where as usual, the γ matrices are Dirac representations of the generators of C^1, $\gamma^a = \rho(\varepsilon^a)$, and they generate $\text{Spin}(1, 3)^+ = SL(2, \mathbb{C})$. Not all γ matrices are unitary and they are said to carry a nonunitary representation of the Lorentz group $SO(1, 3)^+$.

When using the standard model of particle physics in a curved spacetime manifold,[8] the definition of γ_5 has to be generalized to

$$\gamma_5 \equiv \frac{\omega_{[\mu\nu\lambda\kappa]}}{4!}\, e_a^\mu\, e_b^\nu\, e_c^\lambda\, e_d^\kappa\, \gamma^a\gamma^b\gamma^c\gamma^d,$$

where $e_a^\mu \in SO(1, 3)^+$ are the vierbeins, $\omega_{[\mu\nu\lambda\rho]}$ is a four-form, and the γ matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I, \tag{7}$$

$$\{\gamma_5, \gamma^\mu\} = 0. \tag{8}$$

Let $m \in U_i$, where U_i is a chart, whose coordinate is $x^\mu(m)$. The four-form can be expanded as

$$\omega_{[\mu\nu\lambda\kappa]} = \omega_4 = \omega_{0123}(m)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{9}$$

with a positive definite $\omega_{0123} \in \mathcal{F}(U_i)$. So on U_i the four-form is defined and belongs to $\Omega^4(U_i)$. At m, T_mU_i has natural basis $\{\frac{\partial}{\partial x^\mu}\}$ on U_i and a frame $\hat{e} = \{e_0, e_1, e_2, e_3\}$ at m is expressed as

$$e_a = e_a^\mu \left.\frac{\partial}{\partial x^\mu}\right|_m, \quad 0 \leq a \leq 3. \tag{10}$$

[8] Really, ψ is a section of the product bundle $\Gamma(M, (D, M, \pi, \mathbb{C}^4, SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})})) \otimes E$, where E is an associated vector bundle of $P(M, G)$ in an appropriate representation, describing the "inner" degrees of freedom (color etc.). But we can ignore these "inner," gauge, degrees of freedom for now and concentrate on the spacetime transformation properties of the fermionic fields.

In curved space we can therefore locally define the chirality operator $\gamma_{5,i}$ and the projection operators R_i and L_i, whose actions on a section ψ_i of the local Dirac bundle $(D, \pi_D, U_i, \mathbb{C}^4, SL(2,\mathbb{C}) \oplus \overline{SL(2,\mathbb{C})})$ are

$$
\begin{aligned}
L_i \psi_i &= \frac{1}{2}(I - i\gamma_{5,i})\psi_i = \psi_{L,i}, \\
R_i \psi_i &= \frac{1}{2}(I + i\gamma_{5,i})\psi_i = \psi_{R,i},
\end{aligned}
\tag{11}
$$

where $\psi_{L,i}$ becomes a section of the left-handed *Weyl bundle* $(W, \pi_W, U_i, \mathbb{C}^2, SL(2,\mathbb{C}))$ and $\psi_{i,R}$ becomes a section of the right-handed Weyl bundle $(\overline{W}, \pi_{\overline{W}}, U_i, \mathbb{C}^2, \overline{SL(2,\mathbb{C})})$. These spinor bundles can always be defined locally and are globally defined when w_2 is trivial. We will return to this below, and for now simply note that we can locally write the Dirac bundle as the *Whitney-sum bundle* $D = W \oplus \overline{W}$. That is, $D = W \oplus \overline{W}$ is the pullback bundle of $W \times \overline{W}$ by the map $f_i : U_i \to U_i \times U_i$, defined by $f_i(m) = (m, m)$ (embedding in the diagonal).

$$
\begin{array}{ccc}
D = W \oplus \overline{W} & \xrightarrow{\pi_2} & W \times \overline{W} \\
\pi_1 \downarrow & & \downarrow \pi_{\mathrm{w}} \times \pi_{\bar{\mathrm{w}}} \\
M & \xrightarrow{f} & M \times M
\end{array}
$$

On U_i the fibre of the local Dirac bundle is $\mathbb{C}^2 \oplus \mathbb{C}^2$. If we let $\{t_{ij}^{\mathrm{w}}\}$ and $\{t_{ij}^{\bar{\mathrm{w}}}\}$ be the transition functions of W and $\overline{W}$ respectively, then the transition function T_{ij} of $W \oplus \overline{W}$ is the 4×4 matrix

$$
T_{ij}(m) = \begin{pmatrix} t_{ij}^{\mathrm{w}}(m) & 0 \\ 0 & t_{ij}^{\bar{\mathrm{w}}}(m) \end{pmatrix}. \tag{12}
$$

Note that we now employ the notation $\tilde{t}_{ij} = t_{ij}^{W}$ for clarity.

If we try to extend this reduction of the Dirac bundle to all of M we see that we must extend ω_4 throughout all of M. This amounts to the component ω_{0123} remaining positive definite on any chart U_i, and the positive definiteness must be independent of the choice of coordinates. If this can be done then M is orientable and ω_4 is the volume element defined in Section 3.

Let $m \in U_i \cap U_j \cap U_k \cap U_l \neq \emptyset$, and let x^μ, y^μ, z^μ, and u^μ be the coordinates of U_i, U_j, U_k, and U_l respectively.
Then the frame $\hat{e} = \{e_0, e_1, e_2, e_3\}$ at m can now be expressed as

$$
e_a = e_a^\mu \left.\frac{\partial}{\partial x^\mu}\right|_m = \tilde{e}_a^\mu \left.\frac{\partial}{\partial y^\mu}\right|_m = \overline{e}_a^\mu \left.\frac{\partial}{\partial z^\mu}\right|_m = \breve{e}_a^\mu \left.\frac{\partial}{\partial u^\mu}\right|_m,
$$

where (e_a^μ), $(\tilde{e}_a^\mu)$, $(\overline{e}_a^\mu)$, and $(\breve{e}_a^\mu) \in SO(1,3)^+$. Since $e_a^\mu = \frac{\partial x^\mu}{\partial y^\nu}\, \tilde{e}_a^\nu$ we get

$$
t_{ij}(m) = \left(\left(\frac{\partial x^\mu}{\partial y^\nu} \right)_m \right) \in SO(1,3)^+,
$$

and in a similar manner, for example,

$$t_{jk} = \left(\left(\frac{\partial y^\mu}{\partial z^\nu}\right)_m\right), \quad t_{kl} = \left(\left(\frac{\partial z^\mu}{\partial u^\nu}\right)_m\right), \quad \text{and} \quad t_{li} = \left(\left(\frac{\partial u^\mu}{\partial x^\nu}\right)_m\right). \tag{13}$$

So, using the x coordinate we can write $\gamma^\mu = e_a^\mu \gamma^a$. On the intersection $U_i \cap U_j \cap U_k \cap U_l$ we therefore have

$$\begin{aligned} \gamma^\mu &= e_a^\mu \gamma^a \\ &\Leftrightarrow \tilde{\gamma}^\nu = t_{ij}\gamma^\mu = \tilde{e}_a^\nu \gamma^a = t_{ij}\, e_a^\mu \gamma^a. \end{aligned} \tag{14}$$

With this notation, Eq. (9) for $m \in U_i \cap U_j \cap U_k \cap U_l$ becomes

$$\begin{aligned} \omega_4 &= \omega_{0123}(m)\, \frac{\partial x^0}{\partial y^\mu}\, dy^\mu \wedge \frac{\partial x^1}{\partial y^\nu}\, dy^\nu \wedge \frac{\partial x^2}{\partial y^\lambda}\, dy^\lambda \wedge \frac{\partial x^3}{\partial y^\kappa}\, dy^\kappa \\ &= \omega_{0123}(m)\, \det\!\left(\frac{\partial x^\mu}{\partial y^\nu}\right) dy^0 \wedge \cdots \wedge dy^3 \\ &= \omega_{0123}(m)\, \det(t_{ij})\, dy^0 \wedge \cdots \wedge dy^3 \\ &= \omega_{0123}(m)\, \det(t_{ij})\, \det(t_{jk}) dz^0 \wedge \cdots \wedge dz^3 \\ &= \omega_{0123}(m)\, \det(t_{ij})\, \det(t_{jk})\, \det(t_{kl})\, du^0 \wedge \cdots \wedge du^3 \\ &= \omega_{0123}(m)\, \det(t_{ij})\, \det(t_{jk})\, \det(t_{kl})\, \det(t_{li})\, dx^0 \wedge \cdots \wedge dx^3 \end{aligned} \tag{15}$$

The determinants in Eq. (15) are clearly the Jacobians of the coordinate transformations and must not only be positive, they must be 1. This is satisfied if the structure group is $SO(1, 3)^+$ (and in the general n-dimensional case if the structure group for TM is $SO(r, s)$) so that ω_{0123} is independent of the choice of coordinates).

So we see that to define chirality globally we must have a globally defined four-form, which is equivalent to requiring that w_1 is trivial and M orientable. In higher even dimensions, n, a generalization to a globally defined n-form is straightforward (Visser, 1996).

We have thus recovered the known theorem stated below.

Theorem 8. *Chiral fermions exist if and only if spacetime is orientable (and the second Stiefel–Whitney class vanishes).*

So from our discussions in the previous sections, we propose the following theorem.

Theorem 9. *Chiral (or Weyl) fermions exist if and only if the first and second Stiefel–Whitney classes both vanish.*

The remainder of this section will be devoted to proving the following.

Theorem 10. *In an n-dimensional spacetime (n even) chirality is globally defined if and only if* $w_3 \in \check{H}^3(M;\mathbb{Z}_2)$ *is trivial.*

In the course of doing so we also prove the following.

Corollary 1. w_3 *is trivial if and only if* $w_1 = w_2 = 1$.

This, as mentioned earlier, could already be found using Steenrod squares, but can also be done in a "physicist's way" as outlined here.

We can now see what is required for the chirality operators, projection operators, and the Whitney-sum bundle to be defined globally. If we take $R_i = \frac{1}{2}(I + i\gamma_{5,i})$ (the calculations for L_i are similar) we find that we must have

$$R_i = \frac{1}{2}(I + i\gamma_{5,i}) \tag{16}$$

or

$$\begin{aligned}
R_i &= \frac{1}{2}\left(I + i\left(\frac{\omega_{[\mu\nu\lambda\kappa],i}}{4!}\, e_a^{\mu}\, e_b^{\nu}\, e_c^{\lambda}\, e_d^{\kappa}\, \gamma^a\gamma^b\gamma^c\gamma^d\right)\right) \\
&= \frac{1}{2}\left(I + i\left(\det(t_{ij})\frac{\omega_{[\nu\lambda\kappa\mu],j}}{4!} t_{ij}\, \tilde{e}_a^{\nu}\, t_{ij}\, \tilde{e}_b^{\lambda}\, t_{ij}\, \tilde{e}_c^{\kappa}\, t_{ij}\, \tilde{e}_d^{\mu} \gamma^a\gamma^b\gamma^c\gamma^d\right)\right) \\
&= \frac{1}{2}\left(I + i\left(\det(t_{ij})\, \frac{\omega_{[\nu\lambda\kappa\mu],j}}{4!} (t_{ij})^4\, \tilde{e}_a^{\nu}\, \tilde{e}_b^{\lambda}\, \tilde{e}_c^{\kappa}\, \tilde{e}_d^{\mu}\, \gamma^a\gamma^b\gamma^c\gamma^d\right)\right) \\
&= \frac{1}{2}\left(I + i\left(\det(t_{ij})\, \det(t_{jk})\, \frac{\omega_{[\lambda\kappa\mu\nu],k}}{4!}\, (t_{ij})^4 (t_{jk})^4\right.\right. \\
&\quad \left.\left.\times\, \bar{e}_a^{\lambda}\, \bar{e}_b^{\kappa}\, \bar{e}_c^{\mu}\, \bar{e}_d^{\nu}\, \gamma^a\gamma^b\gamma^c\gamma^d\right)\right) \\
&= \frac{1}{2}\left(I + i\left(\det(t_{ij})\, \det(t_{jk})\, \det(t_{kl})\, \frac{\omega_{[\kappa\mu\nu\lambda],l}}{4!} (t_{ij})^4 (t_{jk})^4 (t_{kl})^4\right.\right. \\
&\quad \left.\left.\times\, \breve{e}_a^{\kappa}\, \breve{e}_b^{\mu}\, \breve{e}_c^{\nu}\, \breve{e}_d^{\lambda}\, \gamma^a\gamma^b\gamma^c\gamma^d\right)\right) \\
&= \frac{1}{2}\left(I + i\left(\det(t_{ij}t_{jk}t_{kl}t_{li})\frac{\omega_{[\mu\nu\lambda\kappa],i}}{4!} (t_{ij}t_{jk}t_{kl}t_{li})^4\, e_a^{\mu}\, e_b^{\nu}\, e_c^{\lambda}\, e_d^{\kappa}\, \gamma^a\gamma^b\gamma^c\gamma^d\right)\right).
\end{aligned} \tag{17}$$

(Note the slight abuse of notation.) Since the t_{ij}s obey the cocycle condition $(t_{ij}t_{jk}t_{kl}t_{li})^4 = I$, and as we discussed above, the determinant is also trivial since the transition functions belong to a Special Orthogonal group, we can extend the projection operators to all of M. If the Dirac spin bundle is globally defined it should be possible to write it as the Whitney sum $D = W \oplus \overline{W}$.

The chiral projection operators should act on a section $\psi = \psi_i$ of the Dirac bundle so that

$$\begin{aligned} R\psi_i &= \frac{1}{2}(I + i\gamma_{5,i})\psi_i \\ &= \frac{1}{2}(I + i\gamma_{5,j})\psi_j = \frac{1}{2}(I + i\gamma_{5,j})T_{ij}\psi_i \\ &\vdots \\ &= \frac{1}{2}(I + i\gamma_{5,i})T_{ij}T_{jk}T_{kl}T_{li}\psi_i \\ &= \psi_{R,i} \end{aligned} \tag{18}$$

is well defined. This is the case if the T_{ij}s define the Dirac spin bundle over M, where, as before the transition function T_{ij} of $D = W \oplus \bar{W}$ is a $(\dim F + \dim F') \times (\dim F + \dim F')$ matrix, F being the fibre of W and F' the fibre of $\bar{W}$:

$$T_{ij}(m) = \begin{pmatrix} t^{\mathrm{W}}_{ij}(m) & 0 \\ 0 & t^{\bar{\mathrm{W}}}_{ij}(m) \end{pmatrix}. \tag{19}$$

In the (1+3)-dimensional case, with $F = F' = \mathbb{C}^2$, T_{ij} becomes a 4×4 matrix belonging to $SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})}$, in the general $r + s = n$-dimensional case, $T_{ij} \in \mathrm{Spin}(r, s) \oplus \overline{\mathrm{Spin}(r, s)}$. For the T_{ij}s to define the Dirac spin bundle over M we see from Eq. (18) that they must satisfy

$$\begin{aligned} T_{ij}T_{jk}T_{kl}T_{li} &= I_{n\times n} \\ &\Rightarrow \begin{pmatrix} t^{\mathrm{W}}_{ij}t^{\mathrm{W}}_{jk}t^{\mathrm{W}}_{kl}t^{\mathrm{W}}_{li} & 0 \\ 0 & t^{\bar{\mathrm{W}}}_{ij}t^{\bar{\mathrm{W}}}_{jk}t^{\bar{\mathrm{W}}}_{kl}t^{\bar{\mathrm{W}}}_{li} \end{pmatrix} = \begin{pmatrix} I_{\frac{n}{2}\times\frac{n}{2}} & 0 \\ 0 & I_{\frac{n}{2}\times\frac{n}{2}} \end{pmatrix} \end{aligned} \tag{20}$$

So T_{ij} defines a Dirac spinor bundle over M if t^{W}_{ij} and $t^{\bar{\mathrm{W}}}_{ij}$ define the W bundle and $\bar{W}$ bundle over M respectively. But we already know that this is the case if the $SO(r, s)^+$ bundle lifts to the $\mathrm{Spin}(r, s)^+$ bundle over M, and the obstruction to this is the second Stiefel–Whitney class.

The universal covering map φ sends both $\mathrm{Spin}(r, s)$ and $\overline{\mathrm{Spin}(r, s)}$ to $SO(1, 3)^+$ so that $\varphi(t^{\mathrm{W}}_{ij}) = \varphi(t^{\bar{\mathrm{W}}}_{ij}) = t_{ij}$ and the Čech 2-cochain is defined as

$$\varphi^{-1}(t_{ij}t_{jk}t_{ki}) = f_2(i, j, k)I.$$

When w_2 is trivial we know that

$$t^{\mathrm{W}}_{ij}t^{\mathrm{W}}_{jk}t^{\mathrm{W}}_{ki} = I = t^{\bar{\mathrm{W}}}_{ij}t^{\bar{\mathrm{W}}}_{jk}t^{\bar{\mathrm{W}}}_{ki}, \tag{21}$$

giving us

$$I = t^{\mathrm{W}}_{ij}t^{\mathrm{W}}_{jk}t^{\mathrm{W}}_{ki}I = t^{\mathrm{W}}_{ij}t^{\mathrm{W}}_{jk}t^{\mathrm{W}}_{ki}t^{\mathrm{W}}_{il}t^{\mathrm{W}}_{li} = t^{\mathrm{W}}_{ij}t^{\mathrm{W}}_{jk}t^{\mathrm{W}}_{kl}t^{\mathrm{W}}_{li} \tag{22}$$

as well as

$$I = t^{\bar{w}}_{ij} t^{\bar{w}}_{jk} t^{\bar{w}}_{ki} I = t^{\bar{w}}_{ij} t^{\bar{w}}_{jk} t^{\bar{w}}_{ki} t^{\bar{w}}_{il} t^{\bar{w}}_{li} = t^{\bar{w}}_{ij} t^{\bar{w}}_{jk} t^{\bar{w}}_{kl} t^{\bar{w}}_{li}. \tag{23}$$

We can therefore define the Čech 3-cochain f_3 as

$$f_3(i, j, k, l)I = T_{ij} T_{jk} T_{kl} T_{li} = \begin{pmatrix} f_2(i, j, k)\, t^{w}_{il} t^{w}_{li} & 0 \\ 0 & f_2(i, j, k) t^{\bar{w}}_{il} t^{\bar{w}}_{li} \end{pmatrix} \tag{24}$$

only when w_2 is trivial. If $\check{H}^2(M; \mathbb{Z}_2)$ itself is trivial, then the Dirac spinor structure is unique, and here we can see why. If $\check{H}^2(M; \mathbb{Z}_2)$ only contains one element, there is only one Weyl spinor structure from which to build to the Dirac spinor structure.

f_3 is obviously symmetric and it is trivially closed, $f_3 = I$ if it can be defined, and thus it determines an element $[f_3] = w_3 \in \check{H}^3(M; \mathbb{Z}_2)$. So, as w_1 is an obstruction to orientability and w_2 an obstruction to a Weyl spinor structure, so w_3 is an obstruction to Dirac spinor structure and global chirality.

6. THE REDUCTION OF THE STRUCTURE GROUP

The first two Stiefel–Whitney classes could be expressed as obstructions to changing the structure group. For w_1 it was to the reduction $O(3, 1) \to SO(3, 1)$, which we can further reduce to its connected part $SO(3, 1)^+$, whereas w_2 was the obstruction to the lifting to universal covering group, $SO(3, 1)^+ \to \text{Spin}(3, 1) \simeq SL(2, \mathbb{C})$, which again was extended to $SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})}$, resulting in the Dirac bundle. Now, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, of $SL(2, \mathbb{C})$ can be written as the complexification of $\mathfrak{su}(2)$, $\mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{su}(2) \otimes \mathbb{C}$. It is therefore natural to assume that we can reduce the structure groups from $SL(2, \mathbb{C})$ to $SU(2)$ and that this reduction has something to do with the third Stiefel–Whitney class.

A famous solution to Eq. (7) in four dimensions is the so-called Pauli–Dirac form, utilizing the four 2×2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, a representation of the $SU(2)$ algebra generators. With $\gamma^\mu = g^{\mu\nu}\gamma_\nu$ we can now write the solution to Eq. (7) as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{25}$$

Chirality splits the set of sections in the Dirac bundle $(D, \pi, M, \mathbb{C}^4, SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})})$ into $SL(2, \mathbb{C})$-invariant components according to the eigenvalues of γ_5.

$SL(2, \mathbb{C})$ is the complexification of $SU(2)$, that is $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$. When we now look at Eq. (25) we see that when chirality splits the sections of the Dirac-bundle, not only does the structure group $SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})}$ split into

$SL(2, \mathbb{C})$ and $\overline{SL(2, \mathbb{C})}$, but it reduces even further to two copies of the unitary group $SU(2)$, giving us the Weyl bundle $(W, \pi, M, \mathbb{C}^2, SU(2))$.

This happens by simply considering Chiral (Weyl) spinors. But we can now conclude that when a chiral structure exists (i.e., if γ_5 is defined globally), then the structure group reduces from $SL(2, \mathbb{C})$ to $SU(2)$, and the Dirac equation reduces to the Weyl equation.

7. CONCLUSION AND OUTLOOK

We gave a "physicist's view" of the first three Stiefel–Whitney classes in terms of restrictions imposed by the standard model of particle physics upon the topology of spacetime. Each of the three classes were related to the definition of a certain bundle, to the change of structure group, and to a physical requirement. Our findings are summarized in the table below.

It becomes natural to look for a similar interpretation of the final Stiefel–Whitney class, w_4.

Class	Bundle	Group	Meaning
w_1	Orientation	$O(3, 1) \to SO(3, 1)$	Orientation
w_2	Dirac	$SO(3, 1)^+ \to SL(2, \mathbb{C})$	Spin structure
w_3	Weyl	$SL(2, \mathbb{C}) \to SU(2)$	Chirality

The problem is where to start. If we consider the changes in structure group, it is natural to consider one more reduction (in fact, the last nontrivial one possible), namely $SU(2) \to SO(3)$. This would be a kind of "mirror image" of the second Stiefel–Whitney class construction, since $SU(2)$ is the double covering of $SO(3)$, and this is isomorphic to Spin(3). We therefore suggest to study the chain of groups already mentioned earlier:

$$O(3, 1) \to SO(3, 1) \to SO(3, 1)^+ \to SL(2, \mathbb{C}) \to SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})} \to SU(2) \to SO(3).$$

On the other hand, the only ingredient we have not really used so far is causality. We therefore conjecture the fourth class to be related to causality and to the group $SO(3)$ in the above chain; this is the topic of a sequel paper, which is in progress.

APPENDIX A: THE ČECH COHOMOLOGY

If a function on M is $\mathbb{Z}_2$-valued, totally symmetric, and defined on $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_r}$, then f is a *Čech r-cochain*. If we by $C^r(M; \mathbb{Z}_2)$ denote the multiplicative group of Čech r-cochains we can define the coboundary operator δ, $C^r(M;\mathbb{Z}_2) \to C^{r+1}(M;\mathbb{Z}_2)$, by

$$(\delta f)(i_0, i_1, \ldots, i_r, r_{r+1}) = \prod_{j=0}^{r+1} f(i_0, \ldots, \hat{i}_j, \ldots, i_{r+1}),$$

where the variables with the circumflex (̂) is omitted. The coboundary operator is nil-potent so that $\delta^2 f = 1$, where 1 is the trivial element, as we employ multiplicative notation. The cocycle group $Z^r(M;\mathbb{Z}_2)$ and the coboundary group $B^r(M;\mathbb{Z}_2)$ are then defined as

$$Z^r(M;\mathbb{Z}_2) = \{f \in C^r(M,\mathbb{Z}_2) \mid \delta f = 1\},$$
$$B^r(M;\mathbb{Z}_2) = \{f \in C^r(M;\mathbb{Z}_2) \mid \exists f' \in C^{r-1}(M;\mathbb{Z}_2) : f = \delta f'\}.$$

Now we can define the rth *Čech cohomology group* by

$$\check{H}(M;\mathbb{Z}_2) = \frac{\ker\delta_r}{\mathrm{im}\delta_{r-1}} = \frac{Z^r(M;\mathbb{Z}_2)}{B^r(M;\mathbb{Z}_2)}.$$

APPENDIX B: THE WEYL TENSOR AND THE PETROV CLASSIFICATION

When the Riemann curvature tensor is defined, we can construct new tensors by index contraction such as the Ricci tensor

$$Ric_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu},$$

the scalar curvature

$$\mathcal{R} = g^{\mu\nu}\, Ric_{\mu\nu},$$

and for $n \geq 4$ the *Weyl tensor*

$$C_{\mu\nu\lambda\kappa} = R_{\mu\nu\lambda\kappa} + \frac{1}{n-2}(Ric_{\mu\lambda}g_{\nu\kappa} - Ric_{\nu\lambda}g_{\mu\kappa} + Ric_{\nu\kappa}g_{\mu\lambda} - Ric_{\mu\kappa}g_{\nu\lambda})$$
$$+ \frac{\mathcal{R}}{(n-2)(n-1)}(g_{\mu\lambda}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\lambda}). \quad \text{(A1)}$$

For $\dim M \geq 4$ a necessary and sufficient condition for conformal flatness is that $C = 0$, that is, if $C = 0$ then every $m \in M$ has a chart (U, φ) containing m such that $g_{\mu\nu} = e^{2\sigma}\, \eta_{\mu\nu}$, where $\sigma \in \mathcal{F}(M)$.

To understand the Petrov classification of the Weyl tensor we need a description of all real spacetime vectors and tensors in terms of the tensor algebra over complex two-vectors (spinors), their complex conjugates and their duals. This is a more basic description, since spinors are only describeable in terms of conventional tensors in orientable causal Lorentzian spacetimes (Stewart, 1991).

We will then define the spinor algebra as the two-dimensional vector space S over $\mathbb{C}$, equipped with a symplectic linear structure. That is, on S we have a nondegenerate bilinear skew-symmetric two-form, called the skew-scalar product, so that for $\psi, \varphi \in S$:

$$[\psi, \varphi] = -[\varphi, \psi].$$

With the standard Lorentzian scalar product defined using the metric g, every null vector is self-orthogonal, but with the skew-scalar product every vector is self-orthogonal. And since S is two-dimensional the space of vectors orthogonal to a nonzero vector ψ consists precisely of all vectors proportional to ψ. The skew-scalar product defines a natural isomorphism, since for each $\psi \in S$ we can associate $[\psi, \cdot]$ in the dual space S^* of S, so that $[\psi, \cdot]$ is a linear map $S \to \mathbb{C}$, by $\varphi \to [\psi, \varphi]$.

We can then define a spin basis for S by taking an arbitrary vector $o \in S$ and letting ι be any nonparallel vector. Then we have that $[o, \iota] \neq 0$ and we may normalize ι to set the skew-scalar product equal to 1.

Given such a spin basis we can now define the components of vectors with respect to this basis. If $\psi \in S$ the components ψ^a, $a = 0, 1$ are

$$\psi = \psi^0 o + \psi^1 \iota,$$

where clearly $o^a = [1, 0]$ and $\iota^a = (0, 1)$. We will then use the standard terminology that $\psi^a \in S$ and $\psi_a \in S^*$.

Since the skew-scalar product acts on two copies of S we may identify it with elements of $S^* \times S^*$ and will write $\varepsilon_{ab} = -\varepsilon_{ba}$, so that

$$[\psi, \varphi] = \varepsilon_{ab}\psi^a\varphi^b \in \mathbb{C}.$$

That (o, ι) is a spin basis for S now translates to the elements of the basis satisfying

$$\varepsilon_{ab}o^a o^b = \varepsilon_{ab}\iota^a\iota^b = 0, \qquad \varepsilon_{ab}o^a\iota^b = 1.$$

In the following the notion of symmetrization $(\cdots)$ and anti-symmetrization $[\cdots]$ is applied to the spinor suffices. Since S is two-dimensional any multivalent (multi-indiced) spinor obeys $\Psi_{\cdots[abc]\cdots} = 0$ since at least two of the indices must be zero. If $\Psi_{\cdots ab\cdots}$ is such a multivalent spinor then we can write it as (Stewart, 1991)

$$\Psi_{\cdots ab\cdots} = \Psi_{\cdots(ab)\cdots} + \frac{1}{2}\varepsilon_{ab}\Psi^{c}_{\cdots c\cdots}$$

Thus any spinor $\Psi_{ab\cdots f}$ is the sum of the totally symmetric spinor $\Psi_{a\cdots f}$ and exterior products of εs with totally symmetric spinors of lower valence.

Although S is a vector space over $\mathbb{C}$, $\bar{S} \neq S$ and so also $\bar{S}^* \neq S^*$. We will then use the notation that if $\psi^a \in S$, then $\overline{\psi^a} \equiv \bar{\psi}^{a'} \in \bar{S}$.

If φ^a is a univalent spinor every nowhere-zero null vector V^μ can be written in one of these forms:

$$V^\mu = \pm\varphi^a\overline{\varphi}^{a'}, \quad \varphi^a \in S \text{ and } \overline{\varphi}^a \in \overline{S}.$$

If then $\Psi_{ab\cdots c}$ is totally symmetric then there exist univalent spinors $\alpha_a, \beta_b, \ldots, \gamma_c$ so that

$$\Psi_{ab\cdots c} = \alpha_{(a}\beta_b \cdots \gamma_{c)}.$$

The corresponding real null vectors $\alpha^a\overline{\alpha}^{a'}, \beta^b\overline{\beta}^{b'}, \ldots, \gamma^c\overline{\gamma}^{c'}$ are called the principal null directions of Ψ.

With this we can write the Weyl tensor as

$$\begin{aligned} C_{\mu\nu\lambda\kappa} &= C_{abcda'b'c'd'} \\ &= \Phi_{abcd}\varepsilon_{a'b'}\varepsilon_{c'd'} + \overline{\Phi}_{a'b'c'd'}\varepsilon_{ab}\varepsilon_{cd}, \end{aligned}$$

where Φ is totally symmetric. Then there exist four univalent spinors $\alpha_a, \beta_b, \gamma_c, \delta_d$, the principal null spinors, so that

$$\Phi = \alpha_{(a}\beta_b\gamma_c\delta_{d)}.$$

And therefore four null directions for Φ, namely $\alpha^a\overline{\alpha}^{a'}, \beta^b\overline{\beta}^{b'}, \gamma^c\overline{\gamma}^{c'}$, and $\delta^d\overline{\delta}^{d'}$. There are then six cases to consider:

- *Type I* or {1, 1, 1, 1}, where none of the four principal null directions coincide. This is known as the algebraically general case.
- *Type II* or {2, 1, 1}, where two directions coincide. This and all other cases are algebraically special.
- *Type D* or {2, 2}, where there are two (different) pairs of repeated principal null directions.
- *Type III* or {3, 1}, where three principal null directions coincide.
- *Type N* or {4}, where all four principal null directions coincide.
- *Type O*, which happens for flat space, that is, when spacetime is empty.

The reader more interested in the spinor algebra and symplectic structure should consult, for example, Stewart (1991).

APPENDIX C: THE INDEX OF A TWO-SPHERE

Let S be some two-dimensional surface in M that is not necessarily spacelike, but that is a topological equivalent to a two-sphere,

$$(x^{\mu_1})^2 + (x^{\mu_2})^2 + (x^{\mu_3})^2 = 1,$$

and that may cross itself at isolated points. Such a crossing point will be regarded as representing two distinct points of S itself, each of these points must be treated independently of the other. So S represents a mapping of a two-sphere into M, $S^2 \to M$, and not just the image of the mapping.

If we deform S slightly we obtain another two-sphere S' in M. We know that generally two two-dimensional surfaces in a four-dimensional manifold will only intersect in a region of dimension zero, that is, in points. Now choose S' so that it only intersects S in isolated, nondegenerate points m_k, $k = 1, 2, \ldots, m$, that is, choose S' so that S and S' have no common tangent vectors at their points of intersection.

If we assign an orientation to S we will also get an orientation for S' since it is derived from S. Then at each intersection point m_k the vectors tangent to the oriented surfaces S and S' span the set of all vectors at m_k, and so define an orientation of this four-dimensional vector space. Now define $\iota(m_k)$ to be $+1$ if this orientation is the same as that of M and -1 if it is the opposite.

The *index* of the surface S is defined by

$$I(S) \equiv \sum_{k=1}^{m} \iota(m_k).$$

If we reverse the orientation originally assigned to the surface S, then we also revert the orientation of S' and so the index is unchanged. Note also that the index of S is independent of the way we distorted S to get S', for under any further deformation of S', points of intersection are created in pairs whose values are $+1$ and -1. Finally the index $I(S)$ must be continuous under deformations of S, and since $I(S)$ takes only integral values we can conclude that the index is invariant under such deformations. Note also, that the definition of this index does not involve the metric defined on M.

And alternative definition and one which is intuitively easier to understand (at least for the authors) is the following: Let V^μ be a vector field that is nonvanishing in a neighborhood of S and which is tangent to S only at nondegenerate points. We now define the new two-surface S' as being obtained from S by moving a small distance along the trajectories of V^μ. The intersection points m_k of S and S' now corresponds precisely to the points at which V^μ is tangent to S. Thus, the index $I(S)$ is the number of times (properly counted with regard to sign) that V^μ is tangent to S.

APPENDIX D: CURVATURE AND SPINOR-STRUCTURE

We shall now display a curvature integral over two-spheres and show that if this integral is less than a certain value it will be a sufficient (but not necessary) condition for the existence of spinor-structure. As Geroch points out in his article (Geroch, 1970) this integral represents one of the few situations in which, without imposing any symmetries, the actual curvature of space with an indefinite metric is known to have a bearing on the global structure of the space; the following is basically a recount of his findings.

M is again paracompact, and spacetime-, time- and space-oriented, and S is the two-sphere which is topologically a two-surface. Choose a point $m \in S$ and a one-parameter family of curves on S, $c_s(r)$, where $s \in [0, 1]$ labels the individual curves and $r \in [0, 1]$ is a parameter along each curve.

The curves are constructed so that they all begin and end at $m \in S$ and all other points of S lie at exactly one of the curves, so that the family of curves cover all of S. The curves with $s = 0$ and $s = 1$ are the “zero” curves that remain at m.

Let V^a and U^a denote the tangent vectors to the lines $s =$ constant and $r =$ constant, where V^a and U^a are normalized by the condition:

$$V^a \nabla_a r = U^a \nabla_a s = 1. \tag{A2}$$

As always Greek letters (a, b, c) are dreibein indices ranging over (1, 2, 3) and they label individual spacelike vectors, while Latin indices are tensor indices.

Now choose an arbitrary unit timelike vector field t^a on M. At $m \in S$ choose a frame (a triad) $\{e^a_\mu\}$ of spacelike vectors that together with t^a form an orthonormal frame at m. This is possible because of both space- and time-orientation of M.

For each value of s, the index that labels the individual curves, we transport the frame of dreibeins $\{e^a_\mu\}$ along the curve c_s according to the equation

$$V^b \nabla_b \, e^a_\mu = -t^a \left(e^c_\mu \, V^b \nabla_b t_c\right). \tag{A3}$$

Under this transport of $\{e^a_\mu\}$ they remain orthogonal to the timelike vector field t^a and to each other. When we have transported the frame of vectors back to $m \in S$ we have a new frame of vierbeins whose timelike vector coincides with t^a, but whose spacelike vectors will in general be different from the original frame. Let $R^\nu_\mu(S)$ denote the corresponding rotation matrix:

$$\begin{aligned} e^a_\mu\big|_{r=1;s} &= R^\nu_\mu(i)\, e^a_\nu\big|_{r=0;s=0}, \\ R^\gamma_\mu R^\nu_\gamma &= \delta^\nu_\mu. \end{aligned} \tag{A4}$$

So for each of the curves (for each value of s) we obtain a rotation at m and so we define a curve $R^\nu_\mu(s)$ in the rotation group. For $s = 0$ and $s = 1$, the rotation is just the identity and R^ν_μ then represents a closed curve, beginning and ending at the identity element of the rotation group.

The tangent vector to this closed curve is obtained by taking the derivative of the first equation (A4) with respect to the index s:

$$R_{\mu\lambda} \frac{d}{ds} R^\lambda_\nu = \left(e^a_\mu \, U^b \nabla_b \, e_{\mu\nu}\right)_{r=1;s} = \int_{c_s} ds \, P_{\mu\nu}, \tag{A5}$$

where the integrand

$$P_{\mu\nu} = P_{[\mu\nu]} = V^c \nabla_c \left(e^a_\mu \, U^b \nabla_b \, e_{\mu\nu}\right). \tag{A6}$$

If we expand this equation using Eq. (A3) and the fact that the Lie derivative of U^a with respect to V^a vanishes

$$\mathcal{L}_V U = (V^a \partial_a U^b - U^a \partial_a V^a)\, e_b = 0. \tag{A7}$$

By construction, we get

$$P_{\mu\nu} = 2\, e^a_\mu\, e^b_\nu\, U^{[d} V^{d]} [(\nabla_c t_b)(\nabla_d t_a) + R_{abcd}]. \tag{A8}$$

We can divide the closed curves in the rotation group into two classes: those that may be contracted to a point (such as small loops or a rotation through an angle of 4π), and those that cannot be contracted (such as a rotation through the angle 2π).

If our rotation matrix $R^\nu_\mu(s)$ is of the latter type, then there is an essential 2π twist in our frames on S. In this case it will not be possible to find a frame in a neighborhood of S, and so M will not have spinor structure.

We can characterize the curve $R^\nu_\mu(s)$ in terms of a length, by using the standard invariant metric on the rotation group

$$L \equiv \frac{1}{\sqrt{2}} \int_0^1 \sqrt{\left(\frac{d}{ds} R_{\mu\nu}\right)\left(\frac{d}{ds} R^{\mu\nu}\right)}\, ds. \tag{A9}$$

Whenever this length L is less than 2π the curve $R^\nu_\mu(s)$ may always be contracted to a point. $L = 2\pi$ happens when we rotate exactly 360° about a single axis. We can therefore conclude that there will necessarily be a frame in the neighborhood of S provided $L < 2\pi$.

When $L > 2\pi$ we have a twist in the frame bundle and so M will not have spinor structure. We can obtain an upper bound for this length, characterizing the curves in the rotation group, which is independent of the indices r and s, by first substituting Eq. (A5) into Eq. (A9):

$$\begin{aligned} L &= \frac{1}{\sqrt{2}} \int_0^1 \sqrt{\left(R_{\mu\lambda} \frac{d}{ds} R^{\nu\lambda}\right)\left(R^{\mu\kappa} \frac{d}{ds} R^{\nu\kappa}\right)}\, ds \\ &= \frac{1}{\sqrt{2}} \int_0^1 \sqrt{\left(\int_0^1 dr\, P_{\mu\nu}\right)\left(\int_0^1 dr\, P^{\mu\nu}\right)}\, ds \\ &\leq \frac{1}{\sqrt{2}} \int_0^1 ds \int_0^1 \sqrt{P_{\mu\nu} P^{\mu\nu}}\, dr. \end{aligned} \tag{A10}$$

But we also have that

$$\begin{aligned} P_{\mu\nu} P^{\mu\nu} &= 4(g^{ap} - t^a t^p)(g^{bq} - t^b t^q) U^{[c} V^{d]} U^{[r} V^{s]} \\ &\quad \times ((\nabla_c t_b)(\nabla_d t_a) + R_{abcd})((\nabla_r t_q)(\nabla_s t_p) + R_{pqrs}) \leq 8\, U^{[c} V^{d]} U^{[r} V^{s]} \\ &\quad \times \big((\nabla_c t_b)(\nabla_d t_a)(\nabla_r t^b)(\nabla_s t^a) + R_{abcd} R^{ab}_{rs} - 2 t^b t^p R_{abcd} R^a_{prs}\big). \end{aligned} \tag{A11}$$

If we then substitute Eq. (A11) into Eq. (A10) and introduce the surface element of the sphere S, $dS = U^{[a}V^{b]}\,ds\,dr$, we get

$$L \leq \int_S 2\sqrt{\left(R_{abcd}R^{ab}_{rs} - 2t^b t^p R_{abcd}R^a_{prs}\right) dS^{cd}\,dS^{rs}} + \int_S \sqrt{(\nabla_c t_b)(\nabla_r t^b)(\nabla_d t_a)(\nabla_s t^a)\,dS^{cd}\,dS^{rs}}. \tag{A12}$$

As can be seen this equation still depends on the arbitrary unit timelike vector field t^a since the metric is not positive definite, being of Lorentzian signature. We can eliminate this dependence only formally, by defining $\mathcal{L}(S)$ as

$$\mathcal{L}(S) = \min_{t^a} \int_S 2\sqrt{\left(R_{abcd}R^{ab}_{rs} - 2t^b t^p R_{abcd}R^a_{prs}\right)}\,dS^{cd}\,dS^{rs} + \int_S \sqrt{(\nabla_c t_b)(\nabla_r t^b)(\nabla_d t_a)(\nabla_s t^a)}\,dS^{cd}\,dS^{rs}. \tag{A13}$$

But $\mathcal{L}$ depends on S in a nonlocal way because of the second integral, and it seems as if $\mathcal{L}$ cannot be expressed as a single integral over S. The relationship that Geroch found between spinor structure and curvature is the following:

A sufficient (but not necessary) condition that M have spinor structure is that every two-surface S that is topologically a two-sphere, may be deformed so that $\mathcal{L}(S) < 2\pi$. Note that we cannot conclude that if M possesses spinor structure then all two-surfaces that are topologically two-spheres obey $\mathcal{L}(S) < 2\pi$. We can only conclude that their index is an even integer.

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